Split structures in general relativity and the Kaluza-Klein theories

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Abstract

We construct a general approach to decomposition of the tangent bundle of pseudo-Riemannian manifolds into direct sums of subbundles, and the associated decomposition of geometric objects. An invariant structure $\mathcal{H}^r$ defined as a set of r projection operators is used to induce decomposition of the geometric objects into those of the corresponding subbundles. We define the main geometric objects characterizing decomposition. Invariant nonholonomic generalizations of the Gauss-Codazzi-Ricci's relations have been obtained. All the known types of decomposition (used in the theory of frames of reference, in the Hamiltonian formulation for gravity, in the Cauchy problem, in the theory of stationary spaces, and so on) follow from the present work as special cases when fixing a basis and dimensions of subbundles, and parameterization of a basis of decomposition. Various methods of decomposition have been applied here for the Unified Multidimensional Kaluza-Klein Theory and for relativistic configurations of a perfect fluid. Discussing an invariant form of the equations of motion we have found the invariant equilibrium conditions and their 3+1 decomposed form. The formulation of the conservation law for the curl has been obtained in the invariant form.

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I. INTRODUCTION

Most approaches and formalisms in General Relativity as well as in the multidimensional Unified Theories are connected with decomposition of spaces into direct sums of subspaces and the associated decomposition of geometrical objects. It means that in addition to usual structures (differentiable structure, the metric structure, and so on) one should introduce **a split structure** which induces the decomposition of manifolds. This extra structure determines decomposition of all objects and structures defined on a manifold. Among varieties of formalism of decomposition are the methods aimed to describe frames of reference and observable quantities in the theory of gravity. Similar methods have gained the wide acceptance in a great number of problems. Some of these problems are the canonical formalism for gravitational waves, the Cauchy problem in General Relativity, construction of the Unified Theory of interacting fields, quantization of the gravitational field, the tetradic formalism, the Newman-Penrose formalism, the theory of stationary and axisymmetric gravitational fields, the multidimensional and four-dimensional cosmologies.

Mathematicians and physicists developed methods of decomposition starting mainly from their intrinsic interests. It often took place independently and parallely, so that sometimes the same advances were overlooked and then refound.

The early stage of the development of mathematical technique for decomposition could be seen in the classical theory of hypersurfaces, and in the theory of n-dimensional surfaces imbedded in the n-dimensional Riemannian manifold [1]. Then, in the history of split methods, we can distinguish several ways. In mathematics, at the classical stage, there were constructed coordinate techniques for non-holonomic spaces and subspaces [2], [3]. Owing to the use of the coordinate language such methods were rather cumbersome. In physics split methods were induced by attempts to create the Unified Theory of fields, and, in particular, by appearance of the Kaluza-Klein theory [4], [5]. This led to construction of a $4 + 1$ split method for a 5-dimensional manifold and gave an impulse to study multidimensional Kaluza-Klein theories and multidimensional cosmologies [6]- [9] (see also references in [9]).

Another physical branch of split methods was developed much more later than the original works on a $4 + 1$ split were. This branch has begun, apparently, with the work by Echart [10] and has been completed in the papers [11]- [16]. There have been constructed $(3+1)$, $(2+2)$, $(1+1+2)$ coordinate split methods and their special cases ([17] and references therein).

The other independent direction to construct a split of spaces in General Relativity is connected with the questions of convenience of mathematical representation of the Einstein equations and with the study of these equations' structure. This branch is brought about by needs for construction of the canonical formalism [18], of various projection formalisms in the theory of the stationary and axisymmetric gravitational fields [20]- [22] and for the positing of the Cauchy problem in General Relativity [19].

Unfortunately, many of the works mentioned above, which have already become classical, use different and often inassociated approaches. Moreover a coordinate language applied there, especially in early works, makes it almost impossible to calculate the Einstein equations for some forms of the metric.

New stage of development of split methods is based on modern differential geometry [23], [24]. Its invariant language have become working one in General Relativity [25]. It is

not only a natural language for geometry in the whole, but also a convenient approach to calculations. Obtaining formulae is reduced, the formulae themselves become universal, and all the calculations can easily be made by computer.

The invariant split method was considered in [26] but without any connection with the previous works on a split. Objects introduced formally in this work have no clear geometrical meaning. One of us proposed the general invariant method of an $(n+m)$ split for pseudo-Riemannian manifolds [27]- [30]. There were most approaches to split unified in the works, and the objects introduced there have clear physical and geometrical meaning. For special cases of $(1+4)$, $(1+3)$, $(2+2)$, $(n+4)$ splits, in the coordinate representation, these objects reduce to known physical characteristics of a system [8]- [16].

Multidimensional cosmologies and the Unified Theories imply that a manifold should be decomposed into more than two submanifolds. That is one reason why a split of a manifold requires the most general representation.

The theory of $(n_1 + n_2 + ... + n_r)$ decomposition of a pseudo-Riemannian manifold into the r non-holonomic orthogonal subbundles Σ_a of a dimension n_a $(a = 1, 2, ...r)$ has been constructed in the present work ($n = n_1+n_2+...+n_r$ is dimension of a manifold). The $(n+m)$ and $(n+1)$ forms of invariant decomposition have obtained as consequences. Choosing the projection operators and gauges of a basis of decomposition we construct various special cases. Some applications of this method are considered. Let us emphasize that we don't refer to problems of global geometry, but use its invariant formulations to construct decomposition of spaces into direct sums of subspaces.

Note, that the theory of structures mentioned above has found its further mathematical development, and now is widely known in differential geometry under the name "almost product structure" [26]. The latter can be treated from " G-structure" point of view [31] (see also [32] and references therein). We will follow a more natural approach and use the term "a split structure", which is, in our opinion, more in the spirit of physical conceptions aroused in General Relativity when dealing with $(3 + 1), (2 + 2), (1 + 1 + 2)$ decomposition of space-time.

The plan of the paper is as follows. In SecII we introduce the necessary notations used in differential geometry, and the main definition of the theory, **a split structure** on a pseudo-Riemannian manifold. Then we introduce the metrics and connections induced on the corresponding subbundles as well as the main associated geometrical objects on the subbundles: the tensors of extrinsic curvature and of extrinsic torsion, analogies of the Ricci coefficients of rotation, and the curvature tensor. The invariant non-holonomic generalization of the Gauss-Codazzi-Ricci's relations has been found as various projections of the curvature tensor into every possible subbundle (see Appendix A).

In SecIII the special case of invariant split structure $r = 2$ (when we deal with two subbundles only) is considered. In this case the generalized coefficients of rotation disappear from the curvature tensor, thereby the final formulae become much simpler. In SecIV the invariant formulae in the $(n + 1)$ split form complete the scheme of the invariant split for a pseudo-Riemannian manifold. Further, for any concrete calculations, we must fix projection operators.

In SecV we, for illustration, briefly consider the $(n + m)$ and $(n + 1)$ coordinate decomposition of the manifold with respect to the natural basis $\{\partial_\mu, dx^\nu\}$. In SecVI the method of $(n + m)$ decomposition is constructed with respect to an adopted basis. All the relations

obtained in SecVI are basic ones for the other variants of decomposition in this paper. The final formulae (in Appendixes B,C) can be used as an algorithm to compute the Ricci tensor, the Riemann tensor, the scalar curvature, and the corresponding Lagrangians.

In SecVII we define the canonical parameterization of a basis of decomposition. There have been obtained the main geometric objects with respect to this basis. Various well known special cases, which follow from this parametrization, are discussed in the section. Connections and relations among them are analyzed.

In SecVIII we obtain the decomposition induced by a given family of surfaces. In SecIX we consider the decomposition induced by a group of motion. On the basis of this section's results we construct Lagrangian of the Unified multidimensional Kaluza-Klein Theory (SecX). This decomposition, apart from everything else, serves as a methodical illustration of the possible use of the present method for physical theories.

Finally SecXI deals with the theory of configurations of a perfect fluid. Using the $(3+1)$ canonical parameterization, one can define a one-form of the enthalpy and a two-form of the curl. We have obtained the invariant equations of motion for a perfect fluid. The conservation law for the curl of an isentropic perfect fluid has been obtained in the invariant form. For this fluid, rotating in the stationary gravitational field, we have also deduced the equilibrium conditions and constructed its Lagrangian.

In this work we considered a torsion-free pseudo-Riemannian manifold only, none the less, our approach can be used without principal changes for theories of gravity with non-zero torsion. We see further development of the present theme in the possible expanding of the invariant decomposition to supergravity theories. We, mostly, used notations and definitions of the works [23] - [25].

II. A SPLIT STRUCTURE ON A PSEUDO-RIEMANNIAN MANIFOLD

Let M be a pseudo-Riemannian manifold with the metric g; $T(M) = \bigcup_{p\in M} T_p$ and $T^*(M) = \bigcup_{p\in M} T^*_p$ are the tangent and cotangent bundles over M, where T_p and T^*_p are the corresponding fibres over a point p of M. The objects $X, Y, Z, ... \in T(M)$ and $\alpha, \beta, \omega, df \in T^*(M)$ denote contravariant and covariant vector fields (d is an exterior differential). We shall denote by $\omega(X)$ an inner product of a one-form ω and vector X. The scalar product of two vectors X, Y, and two forms α, β is determined by the metric g

$$X \cdot Y \equiv (X, Y) \equiv g(X, Y); \qquad < \alpha, \beta > \equiv g^{-1}(\alpha, \beta) \tag{2.1}$$

where g^{-1} is the inverse of the metric g.

We need to note that for each vector field $Y \in T(M)$ a dual one-form ω is determined uniquely by $\omega(X) = g(X, Y)$, $\forall X \in T(M)$. From now on we just will write $\omega = g(\ .\ , Y)$. Then the inverse of the metric g is given by

$$g^{-1}(\omega, \alpha) = g^{-1}(g(\ .\ , Y), \alpha) = \alpha(Y), \qquad \forall Y \in T(M), \quad \forall \alpha \in T^*(M) \tag{2.2}$$

so that $Y = g^{-1}(\ .\ , w)$.

A linear operator L on $T(M)$ is a tensor of type $(1,1)$ which acts according to the relation $L \cdot X \equiv L(X) \in T(M)$, $\forall X \in T(M)$. Then

$$(L^T \cdot \omega)(X) = (\omega \cdot L)(X) \equiv \omega(L(X)), \qquad \forall X \in T(M) \tag{2.3}$$

where L^T is a transpose of an operator L.

The product of two linear operators $L \cdot H$ is defined by

$$(L \cdot H) \cdot X = L \cdot (H \cdot X) \in T(M), \qquad \forall X \in T(M). \tag{2.4}$$

An operator H is called a symmetric one if

$$(H \cdot X, Y) = (X, H \cdot Y), \qquad \forall X, Y \in T(M). \tag{2.5}$$

We have to introduce the new notation, **a split**, which denotes decomposition into direct sums. Therefore we shall say that **a split structure** $\mathcal{H}^r$ is introduced on M if the r linear symmetric operators $H^a (a = 1, 2, ...r)$ of a constant rank with the properties

$$H^a \cdot H^b = \delta^{ab} H^b; \qquad \sum_{a=1}^{r} H^a = I, \tag{2.6}$$

where I is the unit operator ($I \cdot X = I, \ \ \forall X \in T(M)$), are defined on $T(M)$.

Now we introduce the notations:

$$\Sigma^a_p \equiv \mathrm{Im}\ H^a_p; \quad (\Sigma^*_a)_p \equiv \mathrm{Im}\ (H^a_p)^T; \quad n_a = \dim \Sigma^a_p = \dim (\Sigma^*_a)_p \tag{2.7}$$

where Im H^a_p is an image of an operator H^a at a point p of M, i.e. $\Sigma^a_p = \{X_p \in T_p \mid H^a \cdot X_p = X_p\}$. It is important that owing to constancy of a rank of the operator H^a, dimension n_a does not depend on a point p of M.

From the definitions presented here we can obtain the decomposition of the tangent and cotangent spaces:

$$T_p = \bigoplus_{a=1}^{r} \Sigma^a_p; \quad T^*_p = \bigoplus_{a=1}^{r} (\Sigma^*_a)_p; \quad \dim\ T_p = \dim\ T^*_p = \sum_{a=1}^{r} n_a \tag{2.8}$$

where the sign $\oplus$ denotes the direct sum. Thus the tensors $\{H^a\}$ are the projection operators, which bring about decomposition of the fibres T_p, T^*_p into the r local subspaces Σ^a_p and $(\Sigma^*_a)_p$ respectively. By the same way, the bundles $T(M)$ and $T^*(M)$ are decomposed into the $(n_1 + n_2 + ... + n_r)$ subbundles Σ^a, Σ^*_a, so that

$$T(M) = \bigoplus_{a=1}^{r} \Sigma^a; \quad T^*(M) = \bigoplus_{a=1}^{r} \Sigma^*_a; \quad \Sigma^a = \bigcup_{p \in M} \Sigma^a_p; \quad \Sigma^*_a = \bigcup_{p \in M} (\Sigma^*_a)_p. \tag{2.9}$$

Then arbitrary vectors, covectors, and metrics are decomposed according to the scheme:

$$X = \sum_{a=1}^{r} X^a, \quad \alpha = \sum_{a=1}^{r} \alpha_a, \quad g = \sum_{a=1}^{r} g^a, \quad g^{-1} = \sum_{a=1}^{r} g_a^{-1} \tag{2.10}$$

where

$$X^a = H^a \cdot X^a = H^a \cdot X; \quad H^b \cdot X^a = 0; \quad X^a \cdot X^b = 0; \quad (a \neq b) \tag{2.11}$$

$$\begin{aligned}
&\alpha_a = \alpha_a \cdot H^a = \alpha \cdot H^a; \quad \alpha_a \cdot H^b = 0; \quad \alpha_a(X^b) = 0 \quad (a \neq b) \\
&g^a(X^a, Y^a) \equiv g(X^a, Y^a); \quad g_a^{-1}(\alpha_a, \beta_a) \equiv g^{-1}(\alpha_a, \beta_a), \\
&\forall X^a, Y^a \in \Sigma^a, \qquad \forall \alpha_a, \beta_a \in \Sigma_a^* \ .
\end{aligned} \tag{2.12}$$

In these relations $\{g^a\}$ are the metrics induced on the subbundles $\{\Sigma^a\}$ of the tangent bundle $T(M)$. Using this scheme we can obtain the decomposition of more complex tensors. We assume that all objects with indices $a, b, ...$ are defined on the associated subbundles $\Sigma^a,\ \Sigma^b, ...$

Let ∇ be an affine (symmetric and compatible with g) connection such that

$$\nabla_X Y - \nabla_Y X = [X, Y], \quad X(Y \cdot Z) = Z \cdot \nabla_X Y + Y \cdot \nabla_X Z \tag{2.13}$$

where $[X, Y] = XY - YX$ is the Lie bracket of two vector fields X and Y, $\nabla_X Y$ is the covariant derivative of Y in the direction X. A consequence of this is that

$$2Z \cdot \nabla_X Y = X(Y \cdot Z) + Y(Z \cdot X) - Z(X \cdot Y) + Z \cdot [X, Y] + Y \cdot [Z, X] - X \cdot [Y, Z]. \tag{2.14}$$

Then the covariant derivative $\nabla_X T$ of a tensor T of type (s, r), where $s = 0, 1$ with respect to X is defined by

$$(\nabla_X T)(Y_1, ...Y_r) = \nabla_X(T(Y_1, ...Y_r)) - \sum_{i=1}^{r} T(Y_1, ...Y_{i-1}, \nabla_X Y_i, Y_{i+1}, ...Y_r). \tag{2.15}$$

The Lie derivative $\mathcal{L}_X T$ of a tensor T with respect to a vector X and the exterior derivative of an r-form Ω are given by:

$$\begin{aligned}
(\mathcal{L}_X T)(Y_1, ...Y_r) &= \mathcal{L}_X(T(Y_1, ...Y_r)) - \sum_{i=1}^{r} T(Y_1, ...Y_{i-1}, \mathcal{L}_X Y_i, Y_{i+1}, ...Y_r) \\
(d\Omega)(Y_0, Y_1, ...Y_r) &= \sum_{i=0}^{r} (-1)^i Y_i(\Omega(Y_0, ..., \hat{Y}_i, ...Y_r)) \\
&\quad + \sum_{0 \leq i < j \leq r} (-1)^{i+j} \Omega([Y_i, Y_j], Y_0, ..., \hat{Y}_i, ..., \hat{Y}_j, ..., Y_r)
\end{aligned} \tag{2.16}$$

where $\mathcal{L}_X Y = [X, Y], \ \ \mathcal{L}_X \varphi = \nabla_X \varphi = (d\varphi)(X) = X\varphi$ for any scalar function φ; the symbol $\hat{}$ means that the associated term is omitted.

The curvature tensor is defined by the formula

$$R(X, Y)Z = (\nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]})Z. \tag{2.17}$$

Using a split structure $\mathcal{H}^r$, the decomposition of ∇ is easily set up:

$$\nabla_X Y = \sum_{a,b,c=1}^{r} \nabla^a_{X^b} Y^c, \qquad \forall X, Y \in T(M). \tag{2.18}$$

In this sum we can distinguish the five different sorts of objects $\{\nabla^a_{X^a} Y^a,\ \ \nabla^a_{X^b} Y^b,\ \ \nabla^a_{X^a} Y^b,$ $\nabla^a_{X^b} Y^a \ \ \nabla^a_{X^b} Y^c\}$ $(a \neq b \neq c)$, which complete the whole of the projected connections. In particular, in this sum the objects

$$\nabla^a_{X^a} Y^a \equiv H^a \cdot \nabla_{X^a} Y^a, \qquad \forall X^a, Y^a \in \Sigma^a \quad (a = 1, 2, ...r) \tag{2.19}$$

define connections $\{\nabla^a\}$ induced on the subbundles $\{\Sigma^a\}$. The object

$$\nabla^a_{X^b} Y^b \equiv H^a \cdot \nabla_{X^b} Y^b \equiv -B^a(X^b, Y^b), \qquad \forall X^b, Y^b \in \Sigma^b \tag{2.20}$$

is the tensor of extrinsic non-holonomicity of the subbundle Σ^b. One can think that the objects

$$\nabla^a_{X^b} Y^c \equiv H^a \cdot \nabla_{X^b} Y^c \equiv -Q^a(X^b, Y^c) \equiv -Q^a_{bc}(X^b, Y^c), \qquad \forall a \neq b \neq c \tag{2.21}$$

define the generalization of the Ricci's coefficients of rotation γ^a_{bc} [34]. In general case they give the objects of rotation Q^a_{bc} of the subbundles Σ^b, Σ^c in the n_a-dimensional direction Σ^a. The other components can be expressed in terms of the introduced objects and the Lie derivative $\mathcal{L}_{X^b} Y^c$ projected into every possible subbundle Σ^a $(a \neq b \neq c)$. Thus, the components $\nabla^a_{X^a} Y^b \equiv H^a \cdot \nabla_{X^a} Y^b$ and $\nabla^a_{X^b} Y^a \equiv H^a \cdot \nabla_{X^b} Y^a$ satisfy the relations

$$Z^a \cdot \nabla^a_{X^a} Y^b = Y^b \cdot B^b(X^a, Z^a) \quad (a \neq b) \tag{2.22}$$

$$Z^a \cdot \nabla^a_{X^b} Y^a = Z^a \cdot \Lambda^a(X^b, Y^a) + X^b \cdot B^b(Y^a, Z^a) \tag{2.23}$$

where

$$\Lambda^a(X^b, Y^c) = [X^b, Y^c]^a \equiv H^a \cdot [X^b, Y^c] \equiv \mathcal{L}^a_{X^b} Y^c \quad (a \neq b \neq c). \tag{2.24}$$

Taking into account the relation (2.14) and the definition (2.21) we have

$$2Z^a \cdot Q^a(X^b, Y^c) = X^b \cdot \Lambda^b(Y^c, Z^a) - Z^a \cdot \Lambda^a(X^b, Y^c) - Y^c \cdot \Lambda^c(Z^a, X^b).$$

The tensor of extrinsic non-holonomicity can be expressed as the sum of symmetric and antisymmetric parts

$$B^a(X^b, Y^b) = S^a(X^b, Y^b) + A^a(X^b, Y^b) \tag{2.25}$$

where $S^a(X^b, Y^b)$ and $A^a(X^b, Y^b)$ define the tensors of extrinsic curvature and extrinsic torsion of subbundle Σ^b in the direction of the subbundle Σ^a. For these objects one can obtain the relations

$$2Z^a \cdot S^a(X^b, Y^b) = (\mathcal{L}_{Z^a} g^b)(X^b, Y^b) \tag{2.26}$$

$$2A^a(X^b, Y^b) = -[X^b, Y^b]^a \equiv -H^a \cdot [X^b, Y^b]. \tag{2.27}$$

It easy can be shown that a connection ∇^a induced on the subbundle Σ^a will be symmetric and compatible with the metric g^a. The projecting of the curvature tensor into the subbundles $\Sigma^a, \Sigma^b, ...$ gives us the nonholonomic generalizations of the Gauss-Codazzi-Ricci's equations. Using the definitions (2.10-2.12), (2.17-2.27) one can obtain all the necessary projections of the curvature tensor (for more details see Appendix A).

III. INVARIANT $(n+m)$ DECOMPOSITION OF A PSEUDO-RIEMANNIAN MANIFOLD

If $r=2$, then there are only two subbundles Σ' and Σ'' of the tangent bundle $T(M)$ and the previous formulae become much simpler. Owing to importance of this case it was deemed worthwhile to consider the split structure independently from SecII [29], [30].

Let H' be a linear idempotent symmetric operator of a constant rank with the property

$$H' \cdot H' = H'. \tag{3.1}$$

We shall say that H' defines **a** $(n+m)$ **split structure** on M if

$$\dim \mathrm{Im} H' = n; \quad \dim \mathrm{Ker} H' = m; \quad \dim M = n+m \tag{3.2}$$

where $\mathrm{Ker} H'$ is a kernel of the operator H'. Since H' is defined, thereby we define the operator H'' such that

$$H'' \cdot H'' = H''; \quad H'' \cdot H' = H' \cdot H'' = 0; \quad H' + H'' = I. \tag{3.3}$$

Therefore H' and H'' are the projection operators which determine the split structure $\mathcal{H}^2$ on M due to the definition (2.6). We have

$$T(M) = \Sigma' \oplus \Sigma''; \quad \Sigma' = \bigcup_{p \in M} \Sigma'_p; \quad \Sigma'_p = \mathrm{Im} H'_p; \quad \Sigma'' = \bigcup_{p \in M} \Sigma''_p; \quad \Sigma''_p = \mathrm{Ker}\ H'_p; \tag{3.4}$$

$$\begin{aligned} X = X' + X''; \quad g = g' + g''; \quad g^{-1} = (g')^{-1} + (g'')^{-1}; \\ X' = H' \cdot X; \quad X'' = H'' \cdot X; \quad X' \cdot X'' = 0; \\ g'(X', Y') = g(X', Y'); \quad g''(X'', Y'') = g(X'', Y''). \end{aligned} \tag{3.5}$$

A connection ∇ is decomposed into the following components: a connection on Σ', and the tensor of extrinsic non-holonomicity of the subbundle Σ', respectively

$$\nabla'_{X'} Y' = H' \cdot \nabla_{X'} Y' \tag{3.6}$$

$$B''(X', Y') = -\nabla''_{X'} Y' = -H'' \cdot \nabla_{X'} Y'. \tag{3.7}$$

Other components of ∇ can be expressed in terms of the components (3.6),(3.7) and the Lie derivatives of two vector fields

$$X' \cdot \nabla'_{Y'} Z'' = Z'' \cdot B''(Y', X') \tag{3.8}$$

$$X' \cdot \nabla'_{Y''} Z' = X' \cdot \mathcal{L}_{Y''} Z' + Y'' \cdot B''(Z', X'). \tag{3.9}$$

The rest of the components of ∇ $\{\nabla''_{X''} Y'',\ \nabla'_{X''} Y'',\ \nabla''_{X''} Y',\ \nabla''_{X'} Y''\}$ may be written out by substituting $X', Y', B', H', ...$ for $X'', Y'', B'', H'', ...$ and vice versa in formulae (3.6)-(3.9). This completes the set of all the eight possible projections of the connection.

The tensor B'' may be expressed as the sum of its symmetric and antisymmetric parts:

$$B''(X',Y') = S''(X',Y') + A''(X',Y') \tag{3.10}$$

$$2Z'' \cdot S''(X',Y') = (\mathcal{L}_{Z''}g')(X',Y'); \qquad 2A''(X',Y') = -H'' \cdot [X',Y'] \tag{3.11}$$

where S'' and A'' are the tensors of extrinsic curvature and torsion respectively. If $A'' = 0$, the subbundle Σ' will be holonomic. It means that the subbundle Σ' is the union of the tangent bundles of an m-parameter family of n-dimensional surfaces $\{M^n(q) \subset M\}$, where $q = \{c^i\} \in D \subset R^m$ parameterizes the surfaces $M^n(q)$, and D is some range of parameters c^i $(i = 1, 2, ..., m)$ in R^m, that is $\Sigma' = \bigcup_{q\in D} T(M^n(q))$. This implies that a covector basis of locally exact one-forms $\{dx^i\}$ exists on the dual subbundle $(\Sigma'')^*$, so that each of the surfaces of $\{M^n(q)\}$ is the intersection of hypersurfaces $x^i = c^i$ $(i = 1, 2, ..., m)$ for some values of $c^i \in D$.

Using the definition of the curvature tensor (2.17) one can find every possible projection of the curvature tensor

$$\begin{aligned} R(X',Y')Z' \cdot V' = {} & R'(X',Y')Z' \cdot V' + B''(X',Z') \cdot B''(Y',V') \\ & - \; B''(Y',Z') \cdot B''(X',V') + 2A''(X',Y') \cdot B''(Z',V'); \end{aligned} \tag{3.12}$$

$$\begin{aligned} R(X',Y')Z' \cdot V'' = {} & V'' \cdot \{(\nabla''_{Y'}B'')(X',Z') - (\nabla''_{X'}B'')(Y',Z')\} \\ & + \; 2Z' \cdot B'(A''(X',Y'),V''); \end{aligned} \tag{3.13}$$

$$\begin{aligned} R(X',Y'')Z' \cdot V'' = {} & (Z' \cdot (\nabla'_{X'}B') + < X' \cdot B', Z' \cdot B' >)(Y'',V'') \\ & + \; (V'' \cdot (\nabla''_{Y''}B'') + < Y'' \cdot B'', V'' \cdot B'' >)(X',Z'); \end{aligned} \tag{3.14}$$

where R' is the curvature tensor of the subbundle Σ'

$$R'(X',Y')Z' \equiv \{\nabla'_{X'}\nabla'_{Y'} - \nabla'_{Y'}\nabla'_{X'} - \nabla'_{[X',Y']'} + 2\mathcal{L}'_{A''(X',Y')}\}Z', \quad \forall X',Y',Z' \in \Sigma' \tag{3.15}$$

where $\mathcal{L}'$ - the Lie derivative projected into the subbundle Σ' $(\mathcal{L}'_X Y \equiv H' \cdot \mathcal{L}_X Y)$. This definition of the curvature tensor, introduced in the works [28]- [30], is the invariant generalization of that introduced in coordinate form in [11]- [13] by analogy with [2]. Note that the latter term in (3.15) is necessary in order that the differential curvature operator $R'(X',Y')$ on Σ' be a linear multiplicative one, or, in other words, R' be a tensor of type (1,3) on non-holonomic subbundle Σ'. In similar fashion this concerns the general case of $\mathcal{H}^r$ split structure (see Appendix A, (A6) for $R^a(X^a,Y^a)Z^a$).

The following expression, with the fixed vectors X', Z', Y'', V'',

$$(< Y'' \cdot B'', V'' \cdot B'' >)(X',Z') \equiv < Y'' \cdot B''(X', \;.\;), V'' \cdot B''(\;.\;, Z') >$$

defines the scalar product of the two one-forms $\alpha \equiv Y'' \cdot B''(X', \;.\;)$ and $\beta \equiv V'' \cdot B''(\;.\;, Z')$ according to (2.1) by the metric $(g')^{-1}$. The covariant derivatives of the tensor B' are given by

$$(\nabla'_{X'}B')(Y'',Z'') = \nabla'_{X'}(B'(Y'',Z'')) - B'(\nabla''_{X'}Y'',Z'') - B'(Y'',\nabla''_{X'}Z'') \tag{3.16}$$

$$(\nabla'_{X''}B')(Y'', Z'') = \nabla'_{X''}(B'(Y'', Z'')) - B'(\nabla''_{X''}Y'', Z'') - B'(Y'', \nabla''_{X''}Z''). \tag{3.17}$$

The relations (3.12)-(3.14) are nonholonomic analogies of the well-known Gauss-Codazzi-Ricci's equations. Other nontrivial projections of the curvature tensor may be written out using the substitution " $'$ " for " $''$ " and vice versa.

In the special case of coordinate representation of $(3+1)$ and $(2+2)$ decomposition, the objects introduced above give us the known tensors [11] - [16], which have clear physical and geometrical meaning.

Let us note that the objects, presented in the work [26] may be expressed in terms of these tensors. For example, the torsion tensor introduced there as the Nijenhuis tensor [24] proved to be equal

$$S_{H'}(X, Y) \equiv [X, Y']' + [X', Y]' - [X', Y'] - [X, Y]' = 2A'(X'', Y'') + 2A''(X', Y')$$

and the tensors $T_X Y$ and $Q_X Y$of the work [26] are given by:

$$T_X Y \equiv \nabla''_{X'}Y' + \nabla'_{X'}Y'' = -B''(X', Y') + g^{-1}(Y'' \cdot B''(X', \ . \ ), \ . \ )$$
$$Q_X Y \equiv \nabla'_{X''}Y'' + \nabla''_{X''}Y' = -B'(X'', Y'') + g^{-1}(Y' \cdot B'(X'', \ . \ ), \ . \ ).$$

They have not any simple interpretation even in the classical case of hypersurfaces in M.

IV. AN INVARIANT $(n+1)$ SPLIT STRUCTURE ON A PSEUDO-RIEMANNIAN MANIFOLD

In this section we give the invariant generalization of $(n+1)$ decomposition of spaces (the monad method [13], [15]) as a special case of $(n+m)$ decomposition when $m = 1$.

Let u be a vector field (field of a monad) on M such that $u \cdot u = \varepsilon = \pm 1$. It gives a one-form ω and projection operators uniquely by the formulae

$$\omega(X) = \varepsilon u \cdot X, \qquad \forall X \in T(M) \tag{4.1}$$

$$H'' = u \otimes \omega; \qquad H' = I - H''. \tag{4.2}$$

The operators satisfy all the necessary relations (3.1)-(3.3) when Σ'' is a one-dimensional subbundle ($m = 1$). The tensor product is denoted by "$\otimes$".

Thus, defining vector or covector fields, u or ω respectively, we, thereby, induce an $(n+1)$ split structure on M. For any vector field X and metric g, this implies

$$X = X' + \omega(X)u, \quad g = g' + \varepsilon\omega \otimes \omega, \quad g^{-1} = (g')^{-1} + \varepsilon u \otimes u \ . \tag{4.3}$$

Hence it is apparent that $X'' = \omega(X)u$ is collinear to u. The metrics $g'' = \varepsilon\omega \otimes \omega$, $(g'')^{-1} = \varepsilon u \otimes u$ and g', $(g')^{-1}$ are the metrics on the subbundles Σ'', $\Sigma^{*''}$ and Σ', $\Sigma^{*'}$, correspondingly. A connection ∇ has the following components:

$$\nabla_{X'}Y' = \nabla'_{X'}Y' - B(X', Y')u; \qquad \nabla_u u = \nabla'_u u = -B'(u, u) \equiv F \tag{4.4}$$

where $B(X',Y') = \omega(B''(X',Y'))$. The latter equation in (4.4) follows from the formula $u \cdot u = \varepsilon = \pm 1$. If we consider a congruence of curves for which the vector u is the tangent vector, then F is the first curvature of this congruence. The tensor B of type (0,2) is the tensor of extrinsic non-holonomicity of the subbundle Σ' and can be written as the sum of its symmetric and antisymmetric parts:

$$B(X',Y') = -\omega(\nabla''_{X'}Y') = \varepsilon S(X',Y') + A(X',Y'), \tag{4.5}$$

where $S(X',Y') = \varepsilon\omega(S''(X',Y'))$, $\quad A(X',Y') = \omega(A''(X',Y'))$ and

$$2S(X',Y') = (\mathcal{L}_u g')(X',Y'); \qquad 2A(X',Y') = (d\omega)(X',Y') \tag{4.6}$$

are the tensors of extrinsic curvature and extrinsic torsion of the subbundle Σ'.

The components of the curvature tensor in an $(n+1)$ decomposed form lead to the generalized Gauss-Codazzi-Ricci's equations:

$$\begin{aligned} R(X',Y')Z' \cdot V' = R'(X',Y')Z' \cdot V' &+ \varepsilon[2A(X',Y')B(Z',V') \\ &+ \; B(X',Z')B(Y',V') - B(Y',Z')(X',V')] \end{aligned} \tag{4.7}$$

$$R(X',Y')Z' \cdot u = -2A(X',Y')F \cdot Z' + \varepsilon[(\nabla_{Y'}B)(X',Z') - (\nabla_{X'}B)(Y',Z')] \tag{4.8}$$

$$R(X',u)Y' \cdot u = -Y' \cdot \nabla'_{X'}F + \varepsilon(F \cdot X')(F \cdot Y') + (\varepsilon\mathcal{L}_u B - < B, B^T >)(X',Y') \tag{4.9}$$

where the curvature tensor of the subbundle Σ' (see [27]) is given by

$$R'(X',Y')Z' \equiv \{\nabla'_{X'}\nabla'_{Y'} - \nabla'_{Y'}\nabla'_{X'} - \nabla'_{[X',Y']'} + 2A(X',Y')\mathcal{L}'_u\}Z' \; . \tag{4.10}$$

It is to be noted that for tensors of an arbitrary type the projection operators are constructed by the tensor product of the operators (4.2) and their transposes. If one does no more than $(n+1)$ decomposition of objects only from the Cartan algebra of exterior forms on M then the universal invariant construction of the projection operators is feasible (see for example [33] for $(3+1)$ decomposition).

V. $(n+m)$ DECOMPOSITION OF A PSEUDO-RIEMANNIAN MANIFOLD IN COORDINATE FORM

In order to obtain a coordinate form of the invariant objects it is necessary to choose coordinate covector and vector bases $\{\partial_\mu = \partial/\partial x^\mu\}$, $\{dx^\mu\}$ in the domain U of some map x^μ $(\mu, \nu, \rho, ... = 1, 2, ...n, n+1, ...n+m)$. Then we can find all the relations given above with respect to this basis, i.e. in covariant form.

Thus in the case of an $(n+m)$ decomposition one has

$$\begin{gathered} H' = h'^{\nu}_{\ \mu}\partial_\nu \otimes dx^\mu = h'^{\nu}_{\ \mu}\partial'_\nu \otimes d'x^\mu; \quad \partial'_\mu \equiv h'^{\nu}_{\ \mu}\partial_\nu, \quad d'x^\mu \equiv h'^{\mu}_{\ \nu}dx^\nu \\ H'' = h''^{\nu}_{\ \mu}\partial_\nu \otimes dx^\mu = h''^{\nu}_{\ \mu}\partial''_\nu \otimes d''x^\mu; \quad \partial''_\mu \equiv h''^{\nu}_{\ \mu}\partial_\nu, \quad d''x^\mu \equiv h''^{\mu}_{\ \nu}dx^\nu \\ h'^{\nu}_{\ \mu}h'^{\mu}_{\ \rho} = h'^{\nu}_{\ \rho}; \quad h''^{\nu}_{\ \mu}h''^{\mu}_{\ \rho} = h''^{\nu}_{\ \rho}; \quad h'^{\nu}_{\ \mu}h''^{\mu}_{\ \rho} = 0; \quad h'^{\nu}_{\ \mu} + h''^{\nu}_{\ \mu} = \delta^\nu_\mu \end{gathered} \tag{5.1}$$

$$
\begin{aligned}
g = g' + g'' = g'_{\mu\nu} d'x^\mu \otimes d'x^\nu + g''_{\mu\nu} d''x^\mu \otimes d''x^\nu \\
g_{\mu\nu} \equiv \partial_\mu \cdot \partial_\nu = g'_{\mu\nu} + g''_{\mu\nu}; \quad g'_{\mu\nu} = h'^\rho_{\ \mu} h'^\sigma_{\ \nu} g_{\rho\sigma} \quad g''_{\mu\nu} = h''^\rho_{\ \mu} h''^\sigma_{\ \nu} g_{\rho\sigma}
\end{aligned} \tag{5.2}
$$

Further, introducing the definitions

$$
[\partial'_\mu, \partial'_\nu]' \equiv \lambda^{\rho'}_{\mu'\nu'} \partial'_\rho; \quad [\partial'_\mu, \partial''_\nu]' \equiv \lambda^{\rho'}_{\mu'\nu''} \partial'_\rho; \quad [\partial''_\mu, \partial''_\nu]' \equiv -2A^{\rho'}_{\mu''\nu''} \partial'_\rho \tag{5.3}
$$

$$
\nabla'_{\partial'_\mu} \partial'_\nu \equiv L^{\rho'}_{\mu'\nu'} \partial'_\rho; \qquad B' \equiv B^{\rho'}_{\mu''\nu''} \partial'_\rho \otimes d''x^\mu \otimes d''x^\nu; \tag{5.4}
$$

one has

$$
L^{\rho'}_{\mu'\nu'} = d'x^\rho (\nabla_{\partial'_\mu} \partial'_\nu); \qquad B^{\rho'}_{\mu''\nu''} = dx'^\rho (\nabla_{\partial''_\mu} \partial''_\nu) = S^{\rho'}_{\mu''\nu''} + A^{\rho'}_{\mu''\nu''} \tag{5.5}
$$

$$
2A^{\rho'}_{\mu''\nu''} = h''^\omega_{\ \mu} h''^\gamma_{\ \nu} (h'^\rho_{\ \gamma,\omega} - h'^\rho_{\ \omega,\gamma}); \qquad 2S_{\rho'\mu''\nu''} = \partial'_\rho g''_{\mu\nu} + g''_{\mu\sigma} \lambda^{\sigma''}_{\nu''\rho'} + g''_{\sigma\nu} \lambda^{\sigma''}_{\mu''\rho'}. \tag{5.6}
$$

Here $h_{,\gamma} \equiv \partial h / \partial x^\gamma$; $\mu', \nu', \rho', ..., \mu'', \nu'', \rho'', ... = 1, 2, ...n, n+1, ...n+m$. The indices " $'$ " and " $''$ " indicate that the corresponding objects are associated with the subbundles Σ' and Σ'' respectively. From the previous formulae it follows that there are the objects which are associated with the both subbundles Σ' and Σ''. For instance, the tensor of extrinsic non-holonomicity $B^{\rho'}_{\mu''\nu''}$ is a contravariant vector on the subbundle Σ', and a covariant tensor of rank 2 on the subbundle Σ''.

The other necessary objects can be found by substituting " $'$ " for " $''$ " and vice versa. Using these formulae we can obtain the Gauss-Codazzi-Ricci's equations in terms of the introduced objects. If $n = m = 2$, our treatment is reduced to the dyad formalism (see [16]).

In the case of an $(n+1)$ split structure (see SecIV), we have $u = u^\mu \partial_\mu$, $(\mu, \nu = 1, 2, ..., n, n+1)$, and

$$
u_\mu u^\mu = \varepsilon = \pm 1, \qquad h''_{\ \mu}{}^{\nu} = \varepsilon u_\mu u^\nu, \qquad h'_{\ \mu}{}^{\nu} = \delta_\mu^{\ \nu} - \varepsilon u_\mu u^\nu
$$

$$
g_{\mu\nu} = \varepsilon u_\mu u_\nu + g'_{\mu\nu}; \qquad g^{\mu\nu} = \varepsilon u^\mu u^\nu + g'^{\mu\nu} \tag{5.7}
$$

$$
g'_{\mu\nu} = h'_{\ \mu}{}^{\alpha} h'_{\ \nu}{}^{\beta} g_{\alpha\beta}; \qquad g'^{\mu\nu} = h'^\mu_{\ \alpha} h'^\nu_{\ \beta} g^{\alpha\beta} \tag{5.8}
$$

$$
\partial'_\mu = h'^\nu_{\ \mu} \partial_\nu; \qquad [\partial'_\mu, \partial'_\nu] = \varepsilon A_{\mu\nu} u; \qquad [\partial'_\mu, u] = -F_\mu u \tag{5.9}
$$

$$
2A_{\mu\nu} = h'^\rho_{,\mu} h'^\sigma_{,\nu} (u_{\rho,\sigma} - u_{\sigma,\rho}); \qquad F_\mu = (u_{\mu,\nu} - u_{\nu,\mu}) u^\nu; \qquad 2S_{\mu\nu} = \mathcal{L}_u g'_{\mu\nu}. \tag{5.10}
$$

Replacing all the objects in (4.7)-(4.9) by these relations we can find the components of the curvature tensor. Furthermore if we consider $(3+1)$ decomposition of a relativistic space-time, our formalism is reduced to the monad method [11]- [15], and to his special gauges. In this case abstract geometrical objects will have an explicit physical meaning. So, one can think of $A_{\mu\nu}$ as the local angular velocity tensor of the frame of reference. The first curvature vector of the congruence F_μ determines the acceleration of the reference body in a given point, and $S_{\mu\nu}$ is the rate of strain tensor [17].

VI. $(n+m)$ DECOMPOSITION WITH RESPECT TO AN ADOPTED BASIS

To find all the relations considered above in an $(n+m)$ decomposed form for some fixed basis is a question of great significance for applications. Coordinate form of $(n+m)$ decomposition considered in SecV is rather cumbersome, and the objects themselves prove to be singular. One of the reasons of this is that the range of indices $\mu', \mu'', ...$ is redundant. Therefore it is more convenient for applications to choose adopted bases of $(n+m)$ decomposition which will eliminate such redundancy. One's choice of one basis or another is dictated by a physical situation, requirements of an interpretation of results, or just by the necessity to use the most comfortable way of calculation. We shall present here the invariant relations of SecIII with respect to an adopted basis of decomposition. Note that in such a form the formulae will be quite feasible for any concrete basis of decomposition. All the known types of decomposition (for torsion free theories) can be obtained as special cases of the present formalism by choosing the corresponding bases. In an $(n+m)$ decomposed form our method is essentially useful for calculation of the Riemann tensor, the Ricci tensor, and the curvature scalar by computer.

We shall now consider two adopted dual bases of decomposition: a vector one $\{E_\mu\} = \{E_a, E_i\}$ on $T(M)$, and a covector basis $\{\theta^\mu\} = \{\theta^a, \theta^i\}$ on $T^*(M)$, where $E_b \in \Sigma' \equiv \Sigma^n$, $E_i \in \Sigma'' \equiv \Sigma^m$; $\theta^a \in \Sigma^{*\prime} \equiv \Sigma^{*n}$; $\theta^i \in \Sigma^{*\prime\prime} \equiv \Sigma^{*m}$. According to (3.4-3.5) we have

$$\theta^a(E_b) = \delta^a_b; \quad \theta^a(E_j) = 0; \quad \theta^i(E_b) = 0; \quad \theta^i(E_k) = \delta^i_k \tag{6.1}$$

$$E_b \cdot E_k = 0, \qquad < \theta^a, \theta^i >= 0 \quad (a, b = 1, 2, ..., n;\, i, k = n+1, n+2, ..., n+m). \tag{6.2}$$

It should be emphasized that the indices $a, b, c, ...$ and $i, j, k, ...$ indicate the subbundles Σ^n, Σ^{*n} and Σ^m, Σ^{*m} respectively. With respect to the basis $\{E_\mu\}, \{\theta^\mu\}$ one has

$$H' = E_a \otimes \theta^a; \quad H'' = E_i \otimes \theta^i; \quad g = g' + g'' = \gamma_{ab}\theta^a \otimes \theta^b + h_{ik}\theta^i \otimes \theta^k \tag{6.3}$$

where $\gamma_{ab} = E_a \cdot E_b$ and $h_{ik} = E_i \cdot E_k$ are the components of the metrics g', g'' induced on the subbundles Σ^n and Σ^m.

Then we introduce the definitions

$$\begin{aligned} \nabla'_{E_a} E_b = L^c_{ab} E_c; \qquad & \nabla''_{E_i} E_j = L^k_{ij} E_k; \\ B'(E_i, E_k) = B^a_{ik} E_a; \qquad & B''(E_a, E_b) = B^i_{ab} E_i \end{aligned} \tag{6.4}$$

$$\begin{aligned} [E_a, E_b]' = \lambda^c_{ab} E_c; \qquad & [E_i, E_j]'' = \lambda^k_{ij} E_k; \\ [E_a, E_i]' = \lambda^b_{ai} E_b; \qquad & [E_i, E_a]'' = \lambda^k_{ia} E_k \end{aligned} \tag{6.5}$$

where L^c_{ab} and L^i_{jl} are the coefficients of connections ∇' induced on Σ^n and ∇'' induced on Σ^m. Similarly B^c_{ik} and B^i_{ab} are the coefficients of the tensors of extrinsic non-holonomicity of the subbundles Σ^m and Σ^n respectively. Using the identity (2.14) one can find

$$\begin{aligned} L^c_{ab} = \triangle^c_{ab} + \gamma^c_{ab}; \qquad & L^i_{jk} = \triangle^i_{jk} + \gamma^i_{jk}; \\ B^a_{ik} = S^a_{ik} + A^a_{ik}; \qquad & B^i_{ab} = S^i_{ab} + A^i_{ab} \end{aligned} \tag{6.6}$$

where

$$2\triangle_{cab} = E_a\gamma_{bc} + E_b\gamma_{ac} - E_c\gamma_{ab}; \qquad 2\gamma_{cab} = \lambda_{cab} + \lambda_{bca} - \lambda_{abc} \tag{6.7}$$

$$\begin{aligned} &2S_{aik} = (\mathcal{L}_{E_a} g'')(E_i, E_k) = E_a h_{ik} + \lambda_{ika} + \lambda_{kia}; \\ &2A^a_{ik} = (d\theta^a)(E_i, E_k); \quad 2A_{aik} = -E_a \cdot [E_i, E_k]. \end{aligned} \tag{6.8}$$

The coefficients $A_{iab},\ S_{iab},\ \gamma_{ijk},\ \triangle_{ijk}$, unwritten here, can be obtained from (6.7)-(6.8) by the replacement $(a, b, c, ... \leftrightarrow i, j, k, ...)$. Adhering to this style here and below we shall write and discuss only those relations which can not be found by the change of indices. We should remind also that the indices $(a, b, c, ...)$ are raised and lowered by the metrics γ^{ab} and γ_{ab}. The curvature tensor and its contractions are presented in Appendix B.

In the special case of $(n + 1)$ decomposition, i.e. when $m = 1$ one has adopted bases $\{E_\mu\} = \{E_a, E\}$, $\{\theta^\mu\} = \{\theta^a, \theta\}$, $(a, b = 1, 2, ...n)$, so that

$$\begin{aligned} &\theta^a(E_b) = \delta_a{}^b; \qquad \theta_a(E) = 0 = \theta(E_a); \\ &\theta(E) = 1; \quad E \cdot E_a = 0; \quad E \cdot E \equiv \varepsilon N^2 \end{aligned} \tag{6.9}$$

where $\{E_a\} \in \Sigma^n;\ \ \theta^a \in \Sigma^{*n}$ and $E \in \Sigma^1;\ \theta \in \Sigma^{*1}$. In this case the projectors $H' = E_a \otimes \theta^a$ and $H'' = E \otimes \theta$ induce the decomposition of the metric

$$g = g' + g'' = \gamma_{ab}\theta^a \otimes \theta^b + \varepsilon N^2 \theta \otimes \theta. \tag{6.10}$$

Then using the relations (6.4)-(6.8), (B1)-(B9) when $i = j = k = 1$ or (4.4)-(5.1) when $u = N^{-1}E,\ \ \omega = N\theta$ we can find all the necessary relations in the $(n+1)$ decomposed form in an adopted basis. Thus, from (4.4) it follows that

$$F = N^{-2}(G - (E \log N)E); \qquad G = \nabla_E E. \tag{6.11}$$

The tensor of extrinsic non-holonomicity of the subbundle Σ^n can be written in the form

$$B(E_a, E_b) = \varepsilon S_{ab} + A_{ab} \equiv \varepsilon N^{-1}\mathcal{B}_{ab}; \qquad 2\mathcal{B}_{ab} = 2D_{ab} + F_{ab} \tag{6.12}$$

where

$$\begin{aligned} &S_{ab} = N^{-1}D_{ab}; \ \ 2D_{ab} = (\mathcal{L}_E g')(E_a, E_b) = E\gamma_{ab} + E_a \cdot [E_b, E] + E_b \cdot [E_a, E]; \\ &2A_{ab} = \varepsilon N^{-1}F_{ab}; \quad F_{ab} = \varepsilon N^2(d\theta)(E_a, E_b) = -E \cdot [E_a, E_b]. \end{aligned} \tag{6.13}$$

Acting in the same way as in the previous sections we can find the generalized Gauss-Codazzi-Ricci's equations (see Appendix C).

VII. CANONICAL PARAMETERIZATION OF AN $(n + m)$ SPLIT STRUCTURE AND ITS SPECIAL CASES

The relations of SecVI are invariant under the transformation of adopted bases:

$$\theta^a = L^a_b e^b; \quad \theta^l = L^l_k e^k; \quad E_a = (L^{-1})_a{}^b e_b; \quad E_i = (L^{-1})_i{}^k e_k. \tag{7.1}$$

where $\{L_b^a\}$ and $\{L_i^k\}$ are $(n \times n)$ and $(m \times m)$ non-singular matrices, and $\{(L^{-1})_a{}^b\}$ and $\{(L^{-1})_i{}^k\}$ are their inverse matrices. Using this property of invariance one can choose, without loss of generality, the simplest basis of decomposition which is useful for applications.

For this purpose we consider the expansion of the covector basis on Σ^{*m} in the domain U of definition of the map x^μ $\quad(\mu = 1, 2, ...n, n+1, ...n+m)$, i.e. $\theta^i = \theta^i_\mu dx^\mu$. Due to the fact that the rank of the $n \times (n+m)$ matrix $\{\theta^i_\mu\}$ is equal to n, there is an $(m \times m)$ non-singular matrix $\{\theta^i_k\}$ as a box in $\{\theta^i_\mu\}$. Then the covectors θ^i can be written in the form: $\theta^i = \theta^i_k dx^k + \theta^i_a dx^a = L^i_k(dx^k + N^k_a dx^a) \equiv L^i_k e^k$ where $L^i_k = \theta^i_k$, $\quad N^k_a = (L^{-1})^k_i \theta^i_a$. Thus the covector basis θ^i goes over into the new covector basis $e^k \in \Sigma^{*m}$. The vector basis on Σ^n can be written similarly as $E_a = E_a{}^\mu \partial_\mu$. From the condition of duality $e^k(E_a) = 0$ it follows that $E_a = (L^{-1})^b_a(\partial_b - N^k_b \partial_k) \equiv (L^{-1})^b_a e_b$, where $(L^{-1})_a{}^b = E_a{}^b$. Thereby we defined the new vector basis $e_b \in \Sigma^n$. The other vector and covector bases ($e^i \in \Sigma^m$ and $e^a \in \Sigma^{*n}$ respectively) are defined by the condition of duality up to $(n \cdot m)$ functions B^a_i. As a result one obtains the following parameterization of the basis of decomposition:

$$
\begin{aligned}
&e^a = dx^a + B^a_i e^i \in \Sigma^{*n}; \qquad && e_a = \partial_a - N^i_a \partial_i \in \Sigma^n;\\
&e^i = dx^i + N^i_a dx^a \in \Sigma^{*m}; \qquad && e_i = \partial_i - B^a_i e_a \in \Sigma^m.
\end{aligned} \tag{7.2}
$$

We shall call this parameterization the canonical one.

If one follows similar procedure beginning with the covector basis $\theta^a \in \Sigma^{*n}$, one will obtain the other canonical parameterization of $(n+m)$ decomposition.

$$
\begin{aligned}
&e^a = dx^a + A^a_i dx^i \in \Sigma^{*n}; \qquad && e_a = \partial_a - M^k_a e_k \in \Sigma^n;\\
&e^i = dx^i + M^i_a e^a \in \Sigma^{*m}; \qquad && e_k = \partial_k - A^a_k \partial_a \in \Sigma^m.
\end{aligned} \tag{7.3}
$$

When some metric g is fixed on M, the functions B^a_i (or M^i_a) can be found from the condition of orthogonality (6.2) in terms of $g_{\mu\nu}$ and N^i_a (or A^b_k). If, otherwise, we fix B^a_i (or M^i_a), then we can obtain the metric for both cases according to (6.3):

$$
\begin{aligned}
&g = \gamma_{ab}(dx^a + B^a_i e^i) \otimes (dx^c + B^c_k e^k) + h_{ik} e^i \otimes e^k\\
&g = \gamma_{ab} e^a \otimes e^b + h_{ik}(e^i + M^i_a e^a) \otimes (e^k + M^k_b e^b).
\end{aligned} \tag{7.4}
$$

With respect to the canonically parameterized basis (7.2), the objects (6.6)-(6.8) and the Lie bracket of the basic vector fields have the form

$$
\begin{aligned}
&\lambda^c_{ab} = -2B^c_i A^i_{ab}; \qquad \lambda^k_{ij} = (B^a_i e_j - B^a_j e_i) N^k_a;\\
&\lambda^c_{ai} = -e_a B^c_i + 2A^k_{ab} B^b_i B^c_k + N^k_{a,i} B^c_k; \qquad \lambda^k_{ia} = -2A^k_{ac} B^c_i - N^k_{a,i}\\
&2A^i_{ab} = e_b N^i_a - e_a N^i_b; \qquad 2A^a_{ij} = e_i B^a_j - e_j B^a_i - \lambda^k_{ij} B^a_k\\
&2S_{aik} = (\mathcal{L}_{e_a} h)(e_i, e_k); \qquad 2S_{iab} = (\mathcal{L}_{e_i} \gamma)(e_a, e_b)
\end{aligned} \tag{7.5}
$$

where $\gamma = \gamma_{ab} e^a \otimes e^b$ and $h = h_{ik} e^i \otimes e^k$. Here all the geometrical characteristics are expressed in terms of the functions $h_{ij}, \gamma_{ab}, B^a_i, N^k_b$ and their derivatives. Substituting the objects (7.5) for those used in (B2)-(B8) we can obtain the Riemann tensor, the Ricci tensor and the scalar curvature in an $(n+m)$ decomposed form with respect to the canonically parameterized basis (7.2). All the relations for the parameterization (7.3) are found from (7.5) by the substitution $(a, b \leftrightarrow i, j;\ \ B^a_i \to M^i_a,\ \ N^i_a \to A^a_i)$.

In the case of $(n+1)$ decomposition both types of parameterizations should be considered independently. Thus for the $(3+1)$ monad method there are two kinds of canonical parameterizations (with respect to local coordinates $\{x^\mu\} = \{t, x^i\}$) determined by

$$\begin{aligned} e_0 &= \partial_t - N^i\partial_i = Nu,; \qquad & e^0 &= dt + B_i e^i = N^{-1}\omega; \\ e_i &= \partial_i - B_i e_0; & e^i &= dx^i + N^i dt \end{aligned} \tag{7.6}$$

and

$$\begin{aligned} e_0 &= \partial_t - M^i e_i = Vu,; \qquad & e^0 &= dt + A_i dx^i = V^{-1}\omega; \\ e_i &= \partial_i - A_i\partial_t; & e^k &= dx^k + M^k e^0 \end{aligned} \tag{7.7}$$

where u is a monad vector, ω is a one-form of time such that $\omega(u) = 1$.

The first set of bases (7.6) is the generalization of the well-known ADM parameterization [18]. In this case the metric has the form

$$ds^2 = N^2(dt + B_j e^j)^2 - h_{ik}e^i e^k, \qquad (e^i = dx^i + N^i dt). \tag{7.8}$$

The second set of bases (7.7) implies that the metric is given by

$$ds^2 = V^2(e^0)^2 - h_{ik}(dx^i + M^i e^0)(dx^k + M^k e^0), \tag{7.9}$$

where $e^0 = dt + A_j dx^j$.

The latter parameterization is the generalization of those often used when describing stationary spaces. It is worth emphasizing that the redundant "degrees of freedom" of the metrics (7.8)-(7.9) may be used to fix a frame of reference or to simplify the Einstein equations. In the theory of stationary configurations, representation (7.9) is useful for examining of solutions, for which a flux of matter and the timelike Killing's vectors are non-collinear (so-called skew solutions [22]).

If B_j vanishes the metric (7.8) goes over into the standard ADM parameterization

$$ds^2 = N^2 dt^2 - h_{ik}(dx^i + N^i dt)(dx^k + N^k dt). \tag{7.10}$$

When M^k vanishes, the metric (7.9) has the form

$$ds^2 = V^2(dt + A_j dx^j)^2 - h_{ik}dx^i dx^k. \tag{7.11}$$

This parameterization is often used when describing stationary spaces. If we take $N^i = 0$ or $A_j = 0$ for the metrics (7.10) and (7.11) respectively, then in both cases we have

$$ds^2 = g_{00}dt^2 - h_{ik}dx^i dx^k. \tag{7.12}$$

This kind of decomposition corresponds to a trivial case when Σ^3 is a family of hypersurfaces, where each of hypersurfaces is orthogonal to the curves $x^i = \text{const}$. This decomposition is invariant under the transformations

$$t = t(t'), \quad x^i = x^i(x'^k). \tag{7.13}$$

The three-dimensional part of these transformations acts uniformly on all the hypersurfaces. Now we shall start, otherwise, from three-dimensional transformations (7.13) which can be extended to the gauge ones by supposing that they depend on time, i.e.

$$t = t(t'), \quad x^i = x^i(t', x'^k). \tag{7.14}$$

These transformations, under which the hypersurfaces $t = \text{const}$ remain unchanged, have been called the kinemetric ones [12]. In order that the decomposition of the metric be invariant with respect to (7.14) we must "make longer" the time derivative $\partial_t \to \partial_t - N^i\partial_i$ (simultaneously we take $dx^i \to dx^i + N^i dt$) by using the gauge vector N^i. Thus it leads to the kinemetric method of decomposition [12], which coincides with the ADM representation [18].

Similarly extending the transformations of time we obtain the chronometric transformations [11]

$$t = t(t', x^{k\prime}), \quad x^i = x^i(x'^k). \tag{7.15}$$

It is obvious that the transformations (7.15) do not change the congruence of world lines $x^i = \text{const}$. These transformations have been taken as a basis for the definition of the frame of reference [11]. "Making longer" the time differential $dt \to dt + A_i dx^i$ (herewith $\partial_i \to \partial_i - A_i\partial_t$) we obtain the chronometric method of decomposition. The transformations (7.15) and (7.14) are the complements of one another and form together the general covariant transformations $x^\mu = x^\mu(x'^\nu)$.

Further generalizations of (7.10) and (7.11) lead to various parameterizations of the monad method. Thus, making longer dt, $dt \to dt + B_i e^i$ ($\partial_i \to \partial_i - B_i e_0$) one has the canonical parameterization (7.6),(7.8). Making longer ∂_t, $\partial_t \to \partial_t - M^i e_i$ ($dx^k \to dx^k + M^k e_0$) one obtains the other canonical parameterization (7.7),(7.9) of the monad method for M^4. "Lengthening" as referred to is connected with extension of the admissible transformations, which are not coordinate but basic ones. The generalization (7.8) of the ADM parameterization is invariant under the transformations:

$$\tilde{e}^k = \alpha_i^k e^i, \qquad \tilde{h}_{ij} = \alpha_i^m \alpha_j^n h_{mn}, \qquad \tilde{B}_j = \alpha_j^k B_k. \tag{7.16}$$

If we write the inverse of the metric (7.9)

$$(\partial_s)^2 = V^{-2}(\partial_t - M^i e_i)^2 - h^{ik} e_i e_k \tag{7.17}$$

then it is easily can be seen that the metric (7.9) is invariant under the transformations

$$\tilde{e}_i = \beta_i^k e_k, \qquad \tilde{h}^{ij} = \beta_m^i \beta_n^j h^{mn}, \qquad \tilde{M}^i = \beta_k^i M^k. \tag{7.18}$$

In (7.16), (7.18) $\{\alpha_k^i\}$ and $\{\beta_k^i\}$ are non-singular matrices depending on a point p of M. From this we can clearly see the role of the parameterizations (7.8), (7.9) as such generalizations of the kinemetric and chronometric methods that the corresponding metrics admit non-holonomic transformations of spatial vector and covector bases (7.16) and (7.18) respectively.

VIII. DECOMPOSITION INDUCED BY A FAMILY OF SURFACES

Let $\{M^m \subset M\}$ be an $n-$parameter family of $m-$dimensional surfaces. One may think of these surfaces as intersections of the hypersurfaces $x^a = \text{const}$ i.e. $M^m = \bigcap_a \{x^a = \text{const}\}$. It is obvious that such a family induces $(n+m)$ decomposition of M. Indeed, there exists the vector basis $e_i = \partial_i$ on $T(M^*)$, $(i = n+1, ..., n+m)$, because of holonomicity of the M^m itself. As a consequence of it, the covector basis on the orthogonal to $T(M^m)$ subbundles Σ^n is a set of one-forms $\{e^a = dx^a\}$. The corresponding dual bases to the bases $\{e_i\}$ and $\{e^a\}$ are determined up to $(n \cdot m)$ functions N^i_a such that

$$
\begin{aligned}
e^a &= dx^a \in \Sigma^{*n}, \qquad e_a = \partial_a - N^i_a \partial_i \in \Sigma^n \\
e^i &= dx^i + N^i_a dx^a \in \Sigma^{*m}, \qquad e_i = \partial_i \in \Sigma^m.
\end{aligned} \tag{8.1}
$$

The functions N^i_a are expressed in terms of the components of the metric g by using the condition of orthogonality $e_a \cdot e_i = 0$. Thus the projection operators and the metric have the form:

$$H' = (\partial_a - N^i_a \partial_i) \otimes dx^a, \qquad H'' = \partial_k \otimes (dx^k + N^k_a dx^a) \tag{8.2}$$

$$g = \gamma_{ab} dx^a \otimes dx^b + h_{ik}(dx^i + N^i_c dx^c) \otimes (dx^k + N^k_d dx^d). \tag{8.3}$$

From the form of the metric (8.3) it can be seen that here we used the special case of canonical parameterization of $(n+m)$ decomposition (7.2) when B^a_i vanishes. In this case the formulae (7.5) become much simpler. Thus, one finds

$$\lambda^c_{ab} = 0; \quad \lambda^k_{ij} = 0; \quad \lambda^c_{ai} = 0; \quad \lambda^k_{ia} = -N^k_{a,i}; \quad A^a_{ij} = 0; \tag{8.4}$$

$$2S_{iab} = \gamma_{ab,i}; \qquad 2A^i_{ab} = e_b N^i_a - e_a N^i_b \tag{8.5}$$

$$2S_{aik} = h_{ik,a} - h_{ik,l} N^l_a - h_{lk} N^l_{a,i} - h_{il} N^l_{a,k} \tag{8.6}$$

$$2L_{cab} = 2\triangle_{cab} = e_a \gamma_{bc} + e_b \gamma_{ca} + e_c \gamma_{ab} \tag{8.7}$$

$$2L_{ijk} = 2\triangle_{ijk} = h_{ij,k} - h_{ik,j} - h_{jk,i}. \tag{8.8}$$

The partial derivatives with respect to coordinates x^i and x^a are denoted here by "$,i$" and "$,a$" respectively. Then, according to (B2)-(B8), one can find the curvature tensor and its contractions.

IX. DECOMPOSITION INDUCED BY A GROUP OF ISOMETRIES

Let M admits a non-transitive group of isometries G^n with the n linearly independent Killing's vectors $\{\xi_a\}$, which satisfy the relations

$$[\xi_a, \xi_b] = C^d_{ab}\xi_d \qquad (a, b, d = 1, 2, ...n) \tag{9.1}$$

where the C^d_{ab} are the structure constants and obey the Jacobi identity $C^c_{[ab}C^f_{d]c} = 0$ and the condition $C^c_{ab} + C^c_{ba} = 0$. In addition, the metric g satisfies the Killing's equations:

$$(\mathcal{L}_{\xi_a} g)(X, Y) = \xi_a(X \cdot Y) - [\xi_a, X] \cdot Y - X \cdot [\xi_a, Y] = 0, \qquad \forall X, Y \in T(M). \tag{9.2}$$

The group G^n decomposes M into a family of $m-$codimensional surfaces $\{M^n\} \subset M$, on which G^n is simply transitive ($\{M^n\}$ are invariant manifolds). Thus, we can say that the group G^n induces $(n + m)$ decomposition of M into the $m-$parameter family of $n-$dimensional surfaces of transitivity. Then the subbundle $\Sigma^n = \bigcup T(M^n)$ is a union of the tangent bundles of the family $\{M^n\}$, and Σ^m is a union of all the $m-$dimensional directions, which are tangent to M and orthogonal to $T(M^n)$.

Now we shall start in the same way as in the previous section. Thus one may think of the surfaces M^n as an intersection of the invariant hypersurfaces $\{x^i = \text{const}\}$, i.e. $M^m = \bigcap_i\{x^i = \text{const}\}$, $(i = n + 1, ...n + m)$. Moreover, one has $dx^i(\xi_a) = \xi_a x^i = 0$. This is obvious that the invariant differential one-forms dx^i can be chosen as a covector basis on the subbundles Σ^{*m}. Then there exists the vector basis $\{\partial_a\} \in T(M^n)$, so that $dx^i(\partial_a) = 0$ and $\xi_a = \xi^b_{(a)}\partial_b$. Having extended these bases to the "complete ones": $\{dx^i\} \to \{dx^\mu\} = \{dx^a, dx^i\} \in T^*(M)$ and $\{\partial_a\} \to \{\partial_\mu\} = \{\partial_a, \partial_i\} \in T(M)$, where $dx^\mu(\partial_\nu) = \delta^\mu_\nu$ and $[\xi_a, \partial_i] = 0$, we can define one-forms ω^a such that

$$\omega^a(\xi_b) = \delta^a_b; \quad \omega^a(\partial_i) = 0; \quad \mathcal{L}_{\partial_i}\omega^a = 0 \tag{9.3}$$

$$\mathcal{L}_{\xi_a}\omega^b = -C^b_{ad}\omega^d; \quad 2d\omega^a = C^a_{bd}\omega^b \wedge \omega^d \ . \tag{9.4}$$

Let us now introduce an auxiliary definition. We shall say that a split structure $\mathcal{H}^2$ is compatible with a group of isometries if the conditions of invariance of $\mathcal{H}^2$ are satisfied, i.e. if

$$\mathcal{L}_{\xi_a}H' = 0, \quad \mathcal{L}_{\xi_a}H'' = 0, \quad (a = 1, 2, ...n). \tag{9.5}$$

Using (6.3) and (9.1) one can easily verify that for the other vector and covector bases $\{E_k\} \in \Sigma^m$ and $\{\theta^a\} \in \Sigma^{*n}$ we have, respectively,

$$\mathcal{L}_{\xi_a}\theta^b = -C^b_{ad}\theta^d; \qquad \mathcal{L}_{\xi_a}E_k = 0. \tag{9.6}$$

To concretize the basis of decomposition we take $\theta^a = \theta^a_\mu dx^\mu$ and $E_i = E^\mu_i\partial_\mu$. Then the conditions of duality $\theta^a(\xi_b) = \delta^a_b$, $\theta^a(E_i) = 0$, $dx^k(E_i) = \delta^k_i$ determine these bases up to $(n \cdot m)$ functions A^a_i. As a result the basis of $(n + m)$ decomposition has the form:

$$\begin{aligned} &\xi_a \in \Sigma^n; \qquad e^a = \omega^a + A^a_i dx^i \in \Sigma^{*n} \\ &e_i = \partial_i - A^a_i\xi_a \in \Sigma^m; \quad dx^k \in \Sigma^{*m}, \quad [\xi_a, e_i] = 0. \end{aligned} \tag{9.7}$$

The projection operators and the metric can be written as

$$H' = \xi_a \otimes (\omega^a + A_i^a dx^i); \qquad H'' = (\partial_i - A_i^a \xi_a) \otimes dx^i \tag{9.8}$$

$$g = g' + g'' = \gamma_{ab}(\omega^a + A_i^a dx^i) \otimes (\omega^b + A_j^b dx^j) + h_{kl} dx^k \otimes dx^l \,. \tag{9.9}$$

From the Killing's equations one finds

$$\xi_a \gamma_{bc} - C_{ab}^d \gamma_{dc} - C_{ac}^d \gamma_{bd} = 0; \quad \xi_a A_i^b - C_{ad}^b A_i^d = 0; \quad \xi_a h_{ik} = 0. \tag{9.10}$$

Using these equations we obtain the main geometrical objects

$$\begin{aligned}
&A''(\xi_a, \xi_b) = 0; \qquad 2A'(e_i, e_k) \equiv F_{ik}^a \xi_a \\
&F_{ik}^a = A_{k,i}^a - A_{i,k}^a + C_{bd}^a A_k^b A_i^d \\
&S'(e_i, e_k) = 0; \qquad 2e_i \cdot S''(e_a, e_b) \equiv 2S_{iab} = e_i \gamma_{ab} \\
&2L_{abc} = C_{cab} + C_{bca} + C_{acb}; \\
&2L_{ijk} = 2\triangle_{ijk} = e_j h_{ik} + e_k h_{ij} - e_i h_{jk}.
\end{aligned} \tag{9.11}$$

In the end, from (B2)-(B8), we can find the curvature tensor, the Ricci tensor and scalar curvature (see Appendix D). When $m = 0$ we come to the case of homogeneous spaces.

X. LAGRANGIANS OF THE UNIFIED MULTIDIMENSIONAL KALUZA-KLEIN THEORIES

The mathematical model we shall use for spaces of the unified theories is the totality of the following objects: a) a connected $(4+n)-$dimensional pseudo-Riemannian C^∞ manifold M^{4+n} with a non-singular metric g on it; b) an $n-$parameter compact group of isometries G^n on M^{4+n} with linearly independent Killing's vectors $\xi_a \in T(M^{4+n})$ for which the structure constants C_{bd}^a satisfy the condition $C_{ad}^a = 0$, $(a, b, d = 4, 5, ... n + 3)$.

The physical space-time $V^4 \equiv M^{4+n}/M^n$ is the quotient space M^{4+n} with respect to the invariant manifolds M^n of the group G^n. The V^4 is described by the components h_{ik} of the metric h, by the set of gauge fields A_i^b and by the multiplet $n(n + 1)/2$ of scalar fields $\varphi_{ab} \equiv -\gamma_{ab}$. All these tensors are obtained under the $(4 + n)$ decomposition of M^{4+n} (see SecIX). The true physical configuration is described not by a single set of fields $\{h_{ik}, A_j^b, \varphi_{cd}\}$, but by a whole equivalence class of such sets; each of them corresponds to some point of the orbit G^n. The signature of the metric g is defined by two conditions: first, the metric h is a Lorentz one, and second, the energy density is positive for obtained Lagrangian of fields $\{A_j^b, \varphi_{cd}\}$. In addition, the metric g satisfies the $(4 + n)-$dimensional variational Hilbert Principle for the functional $S[g]$, i.e.

$$\delta S[g] = \delta \left\{ -\frac{1}{4\pi V} \int R^{(4+n)} \Omega^{(4+n)} \right\} = 0 \tag{10.1}$$

where $R^{(4+n)}$ is the curvature scalar on $M^{(4+n)}$, the $(4 + n)-$form $\Omega^{(4+n)}$ is the volume measure on M^{4+n}, and V is the $n-$dimensional invariant volume of M^n

$$V = \int_{M^n} \omega^4 \wedge \omega^5 \wedge ... \wedge \omega^{4+n} \equiv \int_{M^n} \Omega^{(n)}. \tag{10.2}$$

The conditions $C^a_{ab} = 0$ follow from the requirement that the volume measure $\Omega^{(n)}$ must be invariant. They are necessary for compatibility of the variational Hilbert Principle and homogeneity of M^n with respect to the group of isometries G^n. This restricts the admissible variations of fields $\mathcal{L}_{\xi_a}\delta g = 0$ in (10.1). (The similar situation may be found in the theory of homogeneous models of cosmology [35], [36]).

Using the formulae of SecIX and Appendix D for the metric g in the $(n+4)$ decomposed form

$$g = h_{ik}dx^i \otimes dx^k - \varphi_{ab}(\omega^a + A^a_m dx^m) \otimes (\omega^b + A^b_n dx^n) \tag{10.3}$$

and omitting a divergence of some vector, we obtain

$$S^{(4+n)}[g] = S[\varphi_{ab}, A^a_i, h_{jk}] = \int_{V^4} \sqrt{-h} L d^4x. \tag{10.4}$$

The Lagrangian density is

$$\begin{aligned} \sqrt{-h}L = -\frac{1}{4\pi}\sqrt{\mid h\varphi \mid}\{R^{(4)} + \frac{1}{4}\varphi_{ab}F^a_{ij}F^{bij} \\ + \ (\varphi^{ab}\varphi^{cd} - \varphi^{ac}\varphi^{bd})h^{ik}D_i\varphi_{ab}D_k\varphi_{cd} + U(\varphi_{ab})\} \end{aligned} \tag{10.5}$$

where

$$U(\varphi_{ab}) = \frac{1}{2}\varphi^{cd}C^a_{bc}(C^b_{ad} + \frac{1}{2}\varphi_{ap}\varphi^{bq}C^p_{qd}) \tag{10.6}$$

and

$$D_i\varphi_{ab} = \varphi_{ab,i} - T(A_i)^d_a\varphi_{db} - T(A_i)^d_b\varphi_{ad} \tag{10.7}$$

is the gauge-invariant derivative. The components $T(A)^a_b \equiv C^a_{bd}A^d$ of the matrix $T(A)$ realize the adjoint representation of the group G^n: $[T(A), T(B)] = T([A, B]),\ A = A^a\xi_a,\ B = B^a\xi_a$. Lagrangian of this kind (but with the second derivatives of the fields φ_{ab}) has been obtain in [37].

When $n = 1$ Lagrangian (10.5) reduces to Lagrangian of the 5-dimensional Kaluza-Klein Theory [27]. In the static case of spherical symmetry from $n = 1$ it follows Lagrangian of the simple dynamic system. Its equations can be integrated by separation of variables of the corresponding Hamilton-Jacobi equation. In such a way the solution for the interacting scalar, electromagnetic, and gravitational fields was obtained in [38] within the framework of the Unified 5-dimensional Kaluza-Klein Theory.

XI. RELATIVISTIC CONFIGURATIONS OF A PERFECT FLUID

Let us consider space-time M^4 with the metric g in the $(3+1)$ decomposed form

$$g = V^2e^0 \otimes e^0 - h_{ik}e^i \otimes e^k, \quad g^{-1} = V^{-2}e_0 \otimes e_0 - h^{ik}e_i \otimes e_k \tag{11.1}$$

where g^{-1} is the inverse of the metric g. For the time being, we require the basis of decomposition to be an adopted abstract one (i.e. not concretized). Let the source of the gravitational field described by the metric (11.1) be a perfect fluid with the field of 4-velocities $u = V^{-1}e_0 = d/ds$ which is tangent to the flow lines $x^\mu = x^\mu(s)$. Herewith the mass density ρ obeys the conservation law:

$$\mathrm{div}(\rho u) \equiv (\nabla_{e_\mu}\rho u)(e_\mu) = V^{-1}h^{-1/2}\mathcal{L}''_{e_0}(\rho h^{1/2}) = 0 \tag{11.2}$$

where $\mathcal{L}''_{e_0}$ is the Lie derivative with respect to the basis $\{e_i\}$: $\mathcal{L}''_{e_0}\sqrt{h} = \frac{1}{2}\sqrt{h}h^{ik}(\mathcal{L}_{e_0}h)(e_i, e_k)$. The equation of motion for the fluid follows from the relation:

$$\mathrm{div}T \equiv (\nabla_{e_\mu}T)(e^\mu,\ .\ ) = 0. \tag{11.3}$$

The energy-momentum tensor T is

$$T = \mu V^{-2}e_0 \otimes e_0 + Ph^{ik}e_i \otimes e_k \tag{11.4}$$

where μ is the energy density of the fluid, P is the pressure. Using the thermodynamic relations

$$d\mathcal{H} = Tds + \rho^{-1}dP, \qquad \mathcal{H} = (\mu + P)\rho^{-1} \tag{11.5}$$

one finds the equations of motion

$$(\mathrm{div}T)(e_0) = \rho TV^{-1}uS = -\rho V^{-1}dS/ds = 0 \tag{11.6}$$

$$(\mathrm{div}T)(e_i) = h^{ik}(dP - \rho\mathcal{H}\mathcal{L}_u\omega)(e_k) = 0. \tag{11.7}$$

Here we use the following notations: $\mathcal{H}$ is the enthalpy, S is the entropy, T is the temperature, and ω is the covector of the 4-velocity of the fluid ($\omega = Ve^0$, $\omega(u) = 1$). We introduce "the one-form of the enthalpy θ" and "the two-form of the curl Ω" by

$$\theta = \mathcal{H}\omega = \mathcal{H}Ve^0, \qquad \Omega = d\theta. \tag{11.8}$$

Then the equations of motion (11.6), (11.7) can be expressed as

$$\mathcal{L}_{e_0}\theta = d(\mathcal{H}V) - VTdS. \tag{11.9}$$

Using the formula $\mathcal{L}_{e_0} = i_{e_0}d + di_{e_0}$, where the operator i_{e_0} is defined by the relation $(i_{e_0}\Omega)(Y) = \Omega(e_0, Y)$, $\forall Y \in T(M^4)$, we obtain one more form of the equations of motion

$$i_{e_0}\Omega = -VTdS. \tag{11.10}$$

The condition of integrability of these relations leads to the equations of motion for the curl of a perfect fluid

$$\mathcal{L}_{e_0}\Omega = -d(TV) \wedge dS. \tag{11.11}$$

In the special case $S = \text{const}$ a perfect fluid is isentropic so that the equations for "the one-form of the enthalpy" (11.9) and "the two-form of the curl" (11.10), (11.11) are reduced to the relations:

$$\mathcal{L}_{e_0}\theta = d(\mathcal{H}V) \tag{11.12}$$

$$i_{e_0}\Omega = 0, \qquad \mathcal{L}_{e_0}\Omega = 0. \tag{11.13}$$

It is to be note that the last equation in (11.13) is the condition of integrability of the equation (11.12). Moreover we may regard this condition as an invariant formulation of the theorem [39], which states that the two-form of the curl Ω is constant along the world lines of particles of an isentropic perfect fluid. From the first relation in (11.13) it follows that Ω is singular, i.e. $\Omega(e_0, X) = 0$, $\forall X \in T(M^4)$, and therefore "completely spatial". This implies

$$\Omega = \sum_{i,j} \Omega_{ij} e^i \wedge e^j; \qquad \Omega \wedge \Omega = d\theta \wedge d\theta = 0. \tag{11.14}$$

Since in general case $\theta \wedge d\theta \neq 0$, then according to the theorem Darboux (see, for example [23]) it follows that there exist such functions ξ, η, ζ that $\theta = d\xi + \eta d\zeta$. This representation has been used in [40] to construct a number of families of solutions of the Einstein equations for an isentropic perfect fluid.

Now we shall consider the stationary spaces of General Relativity with a timelike Killing's vector ∂_t. Then the equations (11.6), (11.7), as well as their consequences (11.9)-(11.13), go over into the equilibrium conditions of a perfect fluid. For an isentropic stationary flow they admit completely $3-$dimensional formulation. Indeed, in this case one has

$$\mathcal{L}_{\partial_t} g = 0, \quad \mathcal{L}_{\partial_t} e^\mu = 0, \quad [\partial_t, e_\mu] = 0. \tag{11.15}$$

Then using the parameterization of decomposition (7.7) we deduce that the functions V, A_i, M^k, h_{ik} as well as $\rho, \mu, P, \mathcal{H}$ do not depend on time. We define the vector $\vec{M}$ and covector A on the subbundles $\Sigma'' \equiv \Sigma^3$ by

$$\vec{M} = M^i \partial_i, \qquad A = A_k dx^k. \tag{11.16}$$

In terms of $\vec{M}$ and A the conservation law for mass (11.2) is transformed into the $3-$dimensional equation of continuity of the flow lines

$$\mathrm{div}^{(3)}(\rho\vec{M}) \equiv (\nabla_{e_i}\rho\vec{M})(e^i) = h^{-1/2}\mathcal{L}_{\vec{M}}(\rho h^{1/2}) = 0. \tag{11.17}$$

When $S = \text{const}$ the condition (11.9) may be rewritten in the $3-$dimensional form as well

$$i_{\vec{M}} dA = -d\log(\mathcal{H}V); \quad \vec{M}(\mathcal{H}V) = 0. \tag{11.18}$$

From now on the objects and operations are defined on the $3-$dimensional manifold $t = \text{const}$ with respect to the bases $\{\partial_i\}$ and $\{dx^k\}$. For example: $dA = (1/2)\mathcal{F}_{ik} dx^i \wedge dx^k$, where $\mathcal{F}_{ik} = A_{k,i} - A_{i,k}$. The equilibrium condition (11.18) may be expressed in the form

$$\mathcal{L}_{\vec{M}}A = d\{A(\vec{M}) - \log(\mathcal{H}V)\} \qquad (11.19)$$

showing that the one-form $\mathcal{L}_{\vec{M}}\vec{A}$ is exact. Hence, as the condition of integrability one obtains the conservation 3−dimensional theorem for the curl dA along the 3−dimensional flow lines, i.e.

$$\mathcal{L}_{\vec{M}}dA = 0. \qquad (11.20)$$

In the case of parameterization (7.6) for the stationary spaces the functions V, B_i, N^k, h_{jk} do not depend on time either. By analogy with (11.19) one has

$$\mathcal{L}_{\vec{N}}B = -d\log(\mathcal{H}V) \qquad (11.21)$$

where

$$\vec{N} = N^i\partial_i, \qquad B = B_k dx^k. \qquad (11.22)$$

The condition of integrability gives the conservation theorem for the curl of B

$$\mathcal{L}_{\vec{N}}dB = 0. \qquad (11.23)$$

If one of the two objects A and $\vec{M}$ in (11.19) (or $\vec{N}$ and B in (11.21)) vanishes then the equilibrium condition of an isentropic perfect fluid has the simple form

$$\mathcal{H}V = V(\mu + p)/\rho = k \qquad (11.24)$$

where k is the constant. Thus the Lagrangian of an isentropic perfect fluid in equilibrium is

$$L_m \equiv -V\sqrt{h}P = (k\rho - \mu V)\sqrt{h} = [k - (1+\varepsilon)V]\rho\sqrt{h} \qquad (11.25)$$

where $\varepsilon = \varepsilon(\rho)$ is the internal energy of the fluid and $\mu = \rho(1+\varepsilon)$.

As was noted above, the parameterizations (7.6), (7.7) have spurious degrees of freedom. It means that the vector $\vec{M}$ or covector A in (7.7) can be chosen arbitrarily, by using additional physical reasons. Therefore we have a right to introduce the potential of rotation Ψ_1 by the formula

$$\mathcal{L}_{\vec{M}}A = d(\log\Psi_1). \qquad (11.26)$$

Then the equilibrium condition (11.19) is written as a relation for potentials

$$\Psi_1 HV = C_1\exp(A_i M^i), \qquad C_1 = \text{const} \qquad (11.27)$$

and actually gives us the integral of motion. In another case of the parameterization the equilibrium condition (7.6) can be expressed in the form

$$\mathcal{L}_{\vec{N}}B = d(\log\Psi_2), \qquad \Psi_2 HV = C_2 = \text{const}. \qquad (11.28)$$

Thus the potentials Ψ_1 and Ψ_2 are different from each other by the exponential factor $\exp(A_i M^i)$.

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APPENDIX A: THE GENERALIZED GAUSS-CODAZZI-RICCI'S EQUATIONS

Replacing all the connections in the definition of the curvature tensor (2.17) by their "split representatives" (2.18)-(2.21) we have obtained the invariant non-holonomic generalizations of the Gauss-Codazzi-Ricci's equations:

$$\begin{aligned} R(X^a,Y^a)Z^a\cdot V^a = {} & R^a(X^a,Y^a)Z^a\cdot V^a + \sum_{c\neq a}\{2A^c(X^a,Y^a)\cdot B^c(Z^a,V^a) \\ + {} & B^c(Y^a,V^a)\cdot B^c(X^a,Z^a) - B^c(X^a,V^a)\cdot B^c(Y^a,Z^a)\}, \end{aligned} \tag{A1}$$

$$\begin{aligned} R(X^a,Y^a)Z^a\cdot V^b = {} & V^b\cdot\{(\nabla^b_{Y^a}B^b)(X^a,Z^a) - (\nabla^b_{X^a}B^b)(Y^a,Z^a)\} \\ + {} & 2Z^a\cdot B^a(A^b(X^a,Y^a),V^b) + \sum_{c\neq a,b}\{2Z^a\cdot Q^a(A^c(X^a,Y^a),V^b) \\ + {} & B^c(X^a,Z^a)\cdot Q^c(Y^a,V^b) - B^c(Y^a,Z^a)\cdot Q^c(X^a,V^b)\} \end{aligned} \tag{A2}$$

$$\begin{aligned} R(X^a,Y^b)Z^a\cdot V^b = {} & (Z^a\cdot(\nabla^a_{X^a}B^a) + < X^a\cdot B^a, Z^a\cdot B^a >)(Y^b,V^b) \\ + {} & (V^b\cdot(\nabla^b_{Y^b}B^b) + < Y^b\cdot B^b, V^b\cdot B^b >)(X^a,Z^a) \\ + {} & \sum_{c\neq a,b}\{B^c(X^a,Z^a)\cdot B^c(Y^b,V^b) - Q^c(X^a,V^b)\cdot Q^c(Y^b,Z^a) \\ + {} & V^b\cdot Q^b(\Lambda^c(X^a,Y^b),Z^a)\} \end{aligned} \tag{A3}$$

$$\begin{aligned} R(X^a,Y^b)Z^a\cdot V^d = {} & V^d\cdot\{(\nabla^d_{Y^b}B^d)(X^a,Z^a) - (\nabla^d_{X^a}Q^d)(Y^b,Z^a) \\ - {} & B^d(Y^b,B^b(X^a,Z^a))\} + Z^a\cdot\{B^a(Y^b,Q^b(X^a,V^d)) \\ - {} & B^a(\Lambda^d(X^a,Y^b),V^d)\} + (< Y^b\cdot B^b, V^d\cdot B^d >)(X^a,Z^a) \\ - {} & (< X^a\cdot B^a, V^d\cdot Q^d >)(Y^b,Z^a) - \sum_{c\neq a,b,d}\{Z^a\cdot Q^a(\Lambda^c(X^a,Y^b),V^d) \\ - {} & B^c(X^a,Z^a)\cdot Q^c(Y^b,V^d) + Q^c(X^a,V^d)\cdot Q^c(Y^b,Z^a)\} \end{aligned} \tag{A4}$$

$$\begin{aligned} R(X^a,Y^b)Z^c\cdot V^d = {} & V^d\cdot\{(\nabla^d_{Y^b}Q^d)(X^a,Z^c) - (\nabla^d_{X^a}Q^d)(Y^b,Z^c)\} \\ + {} & B^d(X^a,Q^a(Y^b,Z^c)) - B^d(Y^b,Q^b(X^a,Z^c)) + B^d(\Lambda^c(X^a,Y^b),Z^c)\} \\ + {} & Z^c\cdot\{B^c(Y^b,Q^b(X^a,V^d)) - B^c(X^a,Q^a(Y^b,V^d)) \\ - {} & Q^c(\Lambda^c(X^a,Y^b),V^d)\} + (< Y^b\cdot B^b, V^d\cdot Q^d >)(X^a,Z^c) \\ - {} & (< X^a\cdot B^a, V^d\cdot Q^d >)(Y^b,Z^c) + \sum_{f\neq a,b,c,d}\{Q^f(Y^b,V^d)\cdot Q^f(X^a,Z^c) \\ - {} & Q^f(Y^b,Z^c)\cdot Q^f(X^a,V^d) + V^d\cdot Q^d(\Lambda^f(X^a,Y^b),Z^c)\}. \end{aligned} \tag{A5}$$

In the formula (A1) the curvature tensor R^a of the subbundle Σ^a, introduced in [30], is

$$R^a(X^a, Y^a)Z^a \equiv \{\nabla^a_{X^a}\nabla^a_{Y^a} - \nabla^a_{Y^a}\nabla^a_{X^a} - \nabla^a_{[X^a,Y^a]^a} + 2\sum_{c\neq a}\mathcal{L}^a_{A^c(X^a,Y^a)}\}Z^a. \tag{A6}$$

The covariant derivatives of the values B^d and Q^d are given by

$$(\nabla^b_{X^a}B^b)(Y^a, Z^a) = \nabla^b_{X^a}(B^b(Y^a, Z^a)) - B^b(\nabla^a_{X^a}Y^a, Z^a) - B^b(Y^a, \nabla^a_{X^a}Z^a) \tag{A7}$$

$$(\nabla^d_{X^b}B^b)(Y^a, Z^a) = \nabla^d_{X^b}(B^d(Y^a, Z^a)) - B^d(\nabla^a_{X^b}Y^a, Z^a) - B^d(Y^a, \nabla^a_{X^b}Z^a) \tag{A8}$$

$$(\nabla^d_{X^a}Q^d)(Y^b, Z^c) = \nabla^d_{X^a}(Q^d(Y^b, Z^c)) - Q^d(\nabla^b_{X^a}Y^b, Z^c) - Q^d(Y^b, \nabla^c_{X^a}Z^c). \tag{A9}$$

We also used the definition

$$(< Y^b \cdot B^b, V^d \cdot Q^d >)(X^a, Z^c) \equiv < Y^b \cdot B^b(X^a,\ .\ ), V^d \cdot Q^d(\ .\ , Z^c) > . \tag{A10}$$

When fixing the vectors $X^a,\ Y^b,\ V^d,\ Z^c$, the definition (A10) gives us the scalar product $< \alpha^{ba}, \beta^{dc} >$ of one-forms

$$\alpha^{ba} \equiv Y^b \cdot B^b(X^a,\ .\ ), \qquad \beta^{dc} \equiv V^d \cdot Q^d(\ .\ , Z^c)$$

When $n_a = 1 \quad (a = 1, 2, ...r)$, i.e. when all the subbundles are one-dimensional, the relations obtained here reduce to the r-dimensional variant of the tetradic method's formulae [34].

APPENDIX B: COMPONENTS OF THE CURVATURE TENSOR WITH RESPECT TO AN ADOPTED BASIS FOR $(n+m)$ DECOMPOSITION

Due to the definitions

$$\{E_\mu\} = \{E_a, E_i\}; \quad R(E_\mu, E_\nu)E_\rho \cdot E_\sigma = R_{\sigma\rho\mu\nu}; \quad R(E_\mu, E_\nu)E_\rho = R^\sigma_{\rho\mu\nu}E_\sigma \tag{B1}$$

the generalized Gauss-Codazzi-Ricci's equations (3.12)-(3.15) have the form

$$R_{abcd} = R^{(n)}_{abcd} + 2A^i_{.cd}B_{iba} + B^i_{.cb}B_{ida} + B^i_{.db}B_{ica} \tag{B2}$$

$$R_{ibcd} = B_{icb|d} - B_{idb|c} + 2A^k_{.cd}B_{bki} + B^k_{.db}(B_{cik} - \lambda_{kic}) - B^k_{.cb}(B_{dik} - \lambda_{kid}) \tag{B3}$$

$$R_{ibcj} = B_{bji|c} - B_{icb|j} - B_{bjk}B_{ci.}{}^{k} - B_{icd}B_{jb.}{}^{d} + B_{bki}\lambda^k_{.jc} + B_{bjk}\lambda^k_{.ic} + B_{idb}\lambda^d_{.cj} + B_{icd}\lambda^d_{.bj} \tag{B4}$$

where the curvature tensor of the subbundle Σ^n is defined by its components $R^{(n)}_{abcd}$ according to

$$R^{(n)a}{}_{bcd} = E_cL^a_{db} - E_dL^a_{cb} + L^f_{db}L^a_{cf} - L^f_{cb}L^a_{df} - \lambda^f_{cd}L^a_{fb} + 2A^i_{.cd}\lambda^a_{bi} \tag{B5}$$

(and similarly for the replacement $n \to m$ and $a, b, c, ... \leftrightarrow i, j, k, ...$). Then the components of the Ricci tensor and the curvature scalar have the form

$$R_{bd} = R^{(n)}_{bd} - B^i_{db|i} - S_{b|d} + 2A^i_{ad}A_{ib.}{}^a + 2S_{iad}S^{ia}{}_b$$
$$- \; S_b{}^{ij}S_{dij} + A_{bij}A_d{}^{ij} - S_iB^i_{.db} - B^i_{.da}\lambda^a_{.bi} - B^i_{.ab}\lambda^a_{.di} \tag{B6}$$

$$R_{ia} = B_{ia.}{}^b{}_{|b} + B_{ai.}{}^k{}_{|k} - S_{i|a} - S_{a|i} - 2S^b_{ik}S^k_{ab} - 6A^b_{ik}A^k_{ab}$$
$$+ \; S^k(B_{aik} - \lambda_{kia}) + S^b(B_{iab} - \lambda_{bai}) + B^k_{ab}\lambda_{ki.}{}^b + B^b_{ik}\lambda_{ba.}{}^k \tag{B7}$$

$$R = R^{(n)} - 2S^i{}_{|i} - S^iS_i - S^i_{ab}S^{ab}_{i..} - A^i_{ab}A^{ab}_{i..}$$
$$+ \; R^{(m)} - 2S^a{}_{|a} - S^aS_a - S^a_{ij}S^{ij}_{a.} - A^a_{ij}A_{a..}{}^{ij} \tag{B8}$$

where $S^i = S^i_{ab}\gamma^{ab}$, $S^a = S^a_{ik}h^{ik}$. The signs "$_{|i}$" and "$_{|a}$" denote the covariant derivative with respect to the connections L^k_{mn} and L^a_{bc} in the directions of the vectors E_i and E_a respectively. For example

$$B_{icb|d} = E_dB_{icb} - B_{iab}L^a_{dc} - B_{ica}L^a_{db} \quad (a, b, c \leftrightarrow i, j, k). \tag{B9}$$

The other components of the Ricci tensor and the curvature tensor can be found from (B2)-(B7) by the formal substitution $a, b, c, ...$ for $i, j, k, ...$ and otherwise.

APPENDIX C: COMPONENTS OF THE CURVATURE TENSOR WITH RESPECT TO AN ADOPTED BASIS FOR $(n + 1)$ DECOMPOSITION

The generalized Gauss-Codazzi-Ricci's equations for the metric (6.10) with respect to the basis (6.9) have the form:

$$R_{abcd} = R^{(n)}_{abcd} + \varepsilon N^{-2}(\mathcal{B}_{cb}\mathcal{B}_{da} - \mathcal{B}_{db}\mathcal{B}_{ca} + F_{cd}\mathcal{B}_{ba}) \tag{C1}$$

$$R_{n+1,bcd} = N\{(N^{-1}\mathcal{B}_{cb})_{|d} - (N^{-1}\mathcal{B}_{db})_{|c}\} - \varepsilon N^{-2}G_bF_{cd} \tag{C2}$$

$$R_{n+1,bc,n+1} = N\mathcal{L}_E(N^{-1}\mathcal{B}_{cb}) - \mathcal{B}_{ca}\mathcal{B}_{b.}{}^a + \varepsilon N^{-2}G_bG_c - N^2(N^{-2}G_b)_{|c} \tag{C3}$$

$$R_{bd} = R^{(n)}_{bd} - \varepsilon N^{-2}[N\mathcal{L}_E(N^{-1}\mathcal{B}_{db}) + D\mathcal{B}_{db} + \frac{1}{2}F_{ba}F_{d.}{}^a - 2D_{ba}D_{d.}{}^a]$$
$$+ \varepsilon(N^{-2}G_b)_{|d} - N^{-4}G_bG_d \tag{C4}$$

$$R_{n+1,a} = N[(N^{-1}\mathcal{B}_{a.}{}^b)_{|b} - E_a(N^{-1}D)] - \varepsilon N^{-1}F_{ab}G^b \tag{C5}$$

$$R_{n+1,n+1} = -NE(N^{-1}D) - D_{ab}D^{ab} + \frac{1}{4}F_{ab}F^{ab} + \; N^2(N^{-2}G^a)_{|a} - \varepsilon N^{-2}G_aG^a \tag{C6}$$

$$R = R^{(n)} - 2\varepsilon N^{-1}E(N^{-1}D) - \varepsilon N^{-2}(D^2 + D_{ab}D^{ab} + \frac{1}{4}F_{ab}F^{ab})$$
$$+ 2\varepsilon(N^{-2}G_a)_{|a} - 2N^{-4}G_aG^a. \tag{C7}$$

$$R^{(n)a}{}_{bcd} = E_cL^a_{db} - E_dL^a_{cb} + L^f_{db}L^a_{cf} - L^f_{cb}L^a_{df} - \lambda^f_{cd}L^a_{fb} + \varepsilon N^{-2}F_{cd}\lambda^a_b \tag{C8}$$

were $\lambda^a_b = \theta^a([E_b, E])$ and $R^{(n)} = \gamma^{bd}R^{(n)}_{bd}$; $R^{(n)}_{bd} = R^{(n)a}{}_{bad}$.

APPENDIX D: COMPONENTS OF THE CURVATURE TENSOR FOR A DECOMPOSITION INDUCED BY A GROUP OF ISOMETRIES

The curvature tensor and its contractions with respect to the basis (9.7) for the metric (9.9) have the form:

$$R^{(m+n)}_{dcab} = R^{(n)}_{dcab} + S_{ic[a}S^{i}_{b]d} \tag{D1}$$

$$R^{(m+n)}_{icab} = S^{k}_{c[a}F_{b]ki} + S_{icd}C^{d}_{.ba} + 2S_{id[a}\gamma^{d}_{.b]c} \tag{D2}$$

$$R^{(m+n)}_{ickb} = -S_{ibc;k} + S_{ibd}S_{kc.}{}^{d} + \frac{1}{4}F_{ckj}F_{bi.}{}^{j} - \frac{1}{2}\gamma^{d}_{.bc}F_{dki} \tag{D3}$$

$$R^{(m+n)}_{ajkl} = F_{aj[l;k]} + F_{bj[k}S_{l]a}{}^{b} + F_{bkl}S_{ja}{}^{b} \tag{D4}$$

$$R^{(m+n)}_{ijkl} = R^{(m)}_{ijkl} + \frac{1}{2}F_{ai[k}F^{a}_{.l]j} - \frac{1}{2}F_{aij}F^{a}_{kl} \tag{D5}$$

$$R^{(m)i}_{jlk} = 2e_{[k}\triangle^{i}_{l]j} + 2\triangle^{m}_{j[l}\triangle^{i}_{k]m}, \qquad R^{(n)d}_{.cab} = 2\gamma_{q.[a}{}^{d}\gamma_{b]c.}{}^{q} - C^{qd}_{..c}\gamma_{aqb} \tag{D6}$$

$$R^{(m+n)}_{ab} = R^{(n)}_{ab} - S^{i}_{ab;i} - S^{i}_{ab}S_{i} + 2S^{i}{}_{ac}S_{i}{}^{c}{}_{b} + \frac{1}{4}F_{aij}F_{b}{}^{ij} \tag{D7}$$

$$R^{(m+n)}_{ai} = \frac{1}{2}F_{ai}{}^{k}{}_{;k} + \frac{1}{2}F_{ail}S^{l} + C^{d}_{.ba}S^{b}_{id} - C^{b}_{bd}S_{ia}{}^{d} \tag{D8}$$

$$R^{(m+n)}_{ik} = R^{(m)}_{ik} - S_{(i;k)} - S_{iab}S_{k}{}^{ab} + \frac{1}{2}F_{aij}F^{aj}_{..k} \tag{D9}$$

$$R^{(m+n)} = R^{(n)} + R^{(m)} - 2S^{i}{}_{;i} - S^{i}S_{i} - S^{iab}S_{iab} - \frac{1}{4}F^{a}_{ij}F_{a}{}^{ij}. \tag{D10}$$

Here $R^{(m)} = h^{ik}R^{(m)}_{ik}$; $R^{(m)}_{ik} = R^{(m)l}{}_{ilk}$ and $R^{(n)} = \gamma^{bd}R^{(n)}_{bd}$; $R^{(n)}_{bd} = R^{(n)a}{}_{bad}$. The covariant derivative in the direction of the vector e_k with respect to the connection $\triangle^{i}_{jk}$ is denoted by "$;k$".

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